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# On convergence of Weak Greedy Algorithms<sup>1</sup>

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## 1. INTRODUCTION

This paper is devoted to investigation of Weak Greedy Algorithms (WGA) introduced in [T]. We remind some notations and definitions from the theory of greedy algorithms. Let  $H$  be a real Hilbert space with an inner product  $\langle \cdot, \cdot \rangle$  and the norm  $\|x\| := \langle x, x \rangle^{1/2}$ . We say a set  $\mathcal{D}$  of functions (elements) from  $H$  is a dictionary if each  $g \in \mathcal{D}$  has norm one ( $\|g\| = 1$ ) and  $\overline{\text{span}}\mathcal{D} = H$ . We give now the definition of WGA (see [T]). Let a sequence  $\tau = \{t_k\}_{k=1}^{\infty}$ ,  $0 \leq t_k \leq 1$ , be given.

**Weak Greedy Algorithm.** We define  $f_0^\tau := f$ . Then for each  $m \geq 1$ , we inductively define:

1).  $\varphi_m^\tau \in \mathcal{D}$  is any satisfying

$$|\langle f_{m-1}^\tau, \varphi_m^\tau \rangle| \geq t_m \sup_{g \in \mathcal{D}} |\langle f_{m-1}^\tau, g \rangle|;$$

2).

$$f_m^\tau := f_{m-1}^\tau - \langle f_{m-1}^\tau, \varphi_m^\tau \rangle \varphi_m^\tau;$$

3).

$$G_m^\tau(f, \mathcal{D}) := \sum_{j=1}^m \langle f_{j-1}^\tau, \varphi_j^\tau \rangle \varphi_j^\tau.$$

The following theorem has been proven in [T].

**Theorem A.** Assume

$$(1.1) \quad \sum_{k=1}^{\infty} \frac{t_k}{k} = \infty.$$

Then for any dictionary  $\mathcal{D}$  and any  $f \in H$  we have

$$\lim_{m \rightarrow \infty} \|f - G_m^\tau(f, \mathcal{D})\| = 0.$$

In Section 2 of this paper we prove the following theorem.

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**Theorem 1.** *In the class of monotone sequences  $\tau = \{t_k\}_{k=1}^\infty$ ,  $1 \geq t_1 \geq t_2 \geq \dots \geq 0$ , the condition (1.1) is necessary and sufficient for convergence of Weak Greedy Algorithm for each  $f$  and all Hilbert spaces  $H$  and dictionaries  $\mathcal{D}$ .*

In Section 3 we consider another particular case of sequences  $\tau$ . Let  $\mathcal{N} := \{n_k\}_{k=1}^\infty$ ,  $n_1 < n_2 < \dots$ , be a given subsequence of natural numbers and  $0 < t \leq 1$ . We define

$$\tau(\mathcal{N}, t) := \{t_n \quad : \quad t_{n_k} = t \quad \text{and} \quad t_n = 0, \quad n_k < n < n_{k+1}, \quad k = 1, 2, \dots\}.$$

It is convenient for us to impose some regularity restrictions on  $\mathcal{N}$ . We consider the class  $\mathcal{M}$  of sequences

$$\mathcal{M} := \{\{n_k\}_{k=1}^\infty \quad : \quad n_{k+1} - n_k \geq n_k - n_{k-1}; \quad n_{k+1}n_{k-1} \leq n_k^2, \quad k = 2, \dots\}.$$

We prove in Section 3 the following theorem.

**Theorem 2.** *In the class of sequences  $\tau(\mathcal{N}, t)$ ,  $\mathcal{N} \in \mathcal{M}$ , the condition*

$$(I) \quad \sum_{k=1}^{\infty} \frac{(n_{k+1} - n_k)^{1/2}}{n_k} = \infty$$

*is necessary and sufficient for convergence of Weak Greedy Algorithm for each  $f$  and all Hilbert spaces  $H$  and dictionaries  $\mathcal{D}$ .*

## 2. PROOF OF THEOREM 1

The sufficiency of condition (1.1) for convergence follows from Theorem A. We prove here the necessity part in Theorem 1. Let  $H$  be a Hilbert space with an orthonormal basis  $\{e_j\}_{j=1}^\infty$ . For two elements  $e_i, e_j$ ,  $i \neq j$ , and for a positive number  $t \leq 1/3$  we define the procedure which we call "equalizer" and denote  $E(e_i, e_j, t)$ .

**Equalizer**  $E(e_i, e_j, t)$ . Denote  $f_0 := e_i$  and  $g_1 := \alpha_1 e_i - (1 - \alpha_1^2)^{1/2} e_j$  with  $\alpha_1 := t$ . Then  $\|g_1\| = 1$  and  $\langle f_0, g_1 \rangle = t$ . We define the sequences  $f_1, \dots, f_N$ ;  $g_2, \dots, g_N$ ;  $\alpha_2, \dots, \alpha_N$  inductively:

$$f_n := f_{n-1} - \langle f_{n-1}, g_n \rangle g_n; \quad g_{n+1} := \alpha_{n+1} e_i - (1 - \alpha_{n+1}^2)^{1/2} e_j$$

with  $\alpha_{n+1}$  satisfying

$$\langle f_n, g_{n+1} \rangle = t, \quad n = 1, 2, \dots$$

Let  $f_n = a_n e_i + b_n e_j$  and  $N := N_t$  be the number such that

$$a_{N-1} - b_{N-1} \geq \sqrt{2}t, \quad a_N - b_N < \sqrt{2}t.$$

Then we modify the  $N$ -th step as follows. We take  $g_N := 2^{-1/2}(e_i - e_j)$  and

$$f_N = f_{N-1} - \langle f_{N-1}, g_N \rangle g_N.$$

It is clear that  $a_N = b_N$  and

$$t \leq \langle f_{N-1}, g_N \rangle \leq 2t.$$

We list here the following simple relations

$$a_{n+1} = a_n - t\alpha_{n+1}; \quad b_{n+1} = b_n + t(1 - \alpha_{n+1}^2)^{1/2}, \quad n < N - 1;$$

$$(2.1) \quad a_{n+1} - b_{n+1} = a_n - b_n - t(\alpha_{n+1} + (1 - \alpha_{n+1}^2)^{1/2}), \quad n < N - 1;$$

$$\|f_{n+1}\|^2 = \|f_n\|^2 - t^2, \quad n < N - 1.$$

Relation (2.1) and the inequality  $1 \leq x + (1 - x^2)^{1/2} \leq 2^{1/2}$ ,  $0 \leq x \leq 1$ , imply that

$$(2.2) \quad N \leq 1/t$$

and

$$\|f_N\|^2 \geq \|f_{N-1}\|^2 - 4t^2 \geq \|f\|^2 - t - 3t^2.$$

This gives for  $t \leq 1/3$  that

$$\|f_N\|^2 \geq \|f\|^2 - 2t.$$

It is clear that  $E(e_i, e_j, t)$  is a WGA with regard to the dictionary  $e_i, g_1, g_2, \dots, g_N$  with the "weakness" parameter  $t$ .

Let  $1/3 \geq t_1 \geq t_2 \geq \dots \geq 0$  be such that

$$(2.3) \quad \sum_{k=1}^{\infty} \frac{t_k}{k} < \epsilon$$

with  $\epsilon > 0$  to be chosen later. Then

$$\sum_{s=0}^{\infty} t_{2^s} < 2\epsilon.$$

We define WGA and a dictionary  $\mathcal{D}$  as follows. We begin with  $f := e_1$  and apply  $E(e_1, e_2, t)$ . After  $N_{t_1} \geq 1$  steps we get  $g_1^0, \dots, g_{N_{t_1}}^0$  and

$$f^1 = c_1(e_1 + e_2)$$

with the properties

$$\|f^1\|^2 \geq \|f\|^2 - 2t_1; \quad (c_1)^2 \leq 1/2.$$

We use now  $E(e_1, e_3, t_2)$  and  $E(e_2, e_4, t_2)$ . After  $2N_{t_2} \geq 2$  steps we obtain  $g_1^1, \dots, g_{2N_{t_2}}^1$  and

$$f^2 = c_2(e_1 + \dots + e_4)$$

with the properties

$$\|f^2\|^2 \geq \|f^1\|^2 - 2t_2; \quad (c_2)^2 \leq 2^{-2}.$$

After  $s$  iterations we get

$$f^s = c_s(e_1 + \dots + e_{2^s})$$

and apply  $E(e_i, e_{i+2^s}, t_{2^s})$ ,  $i = 1, 2, \dots, 2^s$ . We make  $2^s N_{t_{2^s}} \geq 2^s$  steps and get  $g_1^s, \dots, g_{2^s N_{t_{2^s}}}^s$  and

$$f^{s+1} = c_{s+1}(e_1 + \dots + e_{2^{s+1}})$$

with the properties

$$\|f^{s+1}\|^2 \geq \|f\|^2 - 2t_1 - 2t_2 - \dots - 2t_{2^s} \geq 1 - 2 \sum_{s=1}^{\infty} t_{2^s} \geq 1 - 4\epsilon.$$

$$(c_{s+1})^2 \leq 2^{-s-1}.$$

Choosing  $\epsilon = \frac{3}{16}$  we see that  $\|f^s\| \geq 1/2$  for all  $s$ .

Thus we get that the WGA with  $\tau$  satisfying (2.3) does not converge for  $f = e_1$  with regard to the dictionary

$$\mathcal{D} = \bigcup_{k \in \mathbb{N}} e_k \cup \bigcup_{s \geq 0; 1 \leq l \leq 2^s N_{t_{2^s}}} g_l^s.$$

We will show now how the general case

$$\sum_{k=1}^{\infty} \frac{t_k}{k} < \infty$$

can be reduced to the case (2.3). We find  $n$  such that

$$\sum_{s=n}^{\infty} t_{2^s} < \epsilon,$$

take  $f = e_1 + \dots + e_{2^n}$  and pick at the first  $2^n - 1$  steps  $e_1, \dots, e_{2^n-1}$  as approximating elements from the dictionary. Then we use the described above procedure with  $f = e_{2^n}$  instead of  $e_1$  with the natural change in indices.

### 3. PROOF OF THEOREM 2

We consider here the case of  $\tau = \{t_n\}_{n=1}^{\infty}$  of the form

$$(3.1) \quad t_{n_k} = t \quad \text{and} \quad t_n = 0, \quad n_k < n < n_{k+1}, \quad k = 1, 2, \dots,$$

for a given subsequence  $n_1 < n_2 < \dots$ . Theorem A implies that the Weak Greedy Algorithm with the above  $\tau$  converges if

$$\sum_{k=1}^{\infty} 1/n_k = \infty.$$

Theorem 2 shows that the above condition can be replaced by a weaker one. We begin with the proof of the sufficiency part of Theorem 2.

**Lemma 3.1.** *In the class  $\mathcal{M}$  of sequences*

$$\mathcal{M} := \{ \{n_k\}_{k=1}^\infty \quad : \quad n_{k+1} - n_k \geq n_k - n_{k-1}; \quad n_{k+1}n_{k-1} \leq n_k^2, \quad k = 2, \dots \}$$

*the following two conditions are equivalent*

$$(I) \quad \sum_{k=1}^{\infty} \frac{(n_{k+1} - n_k)^{1/2}}{n_k} = \infty;$$

$$(II) \quad \forall \{a_j\} \in l_2 \quad \liminf_{k \rightarrow \infty} a_{n_k} \sum_{j=1}^{n_k} a_j = 0.$$

**Remark 3.1.** We point out here that in the proof of  $(I) \Rightarrow (II)$  in Lemma 3.1 we use only the property of boundedness of  $n_{k+1}/n_k$ :

$$(B) \quad \exists C \quad : \quad \forall k \in \mathbb{N}, \quad \frac{n_{k+1}}{n_k} \leq C.$$

Thus in the sufficiency part of Theorem 2 the assumption  $\{n_k\}_{k=1}^\infty \in \mathcal{M}$  can be replaced by a weaker assumption (B). We note also that in the proof of  $(II) \Rightarrow (I)$  in Lemma 3.1 we use only the property of convexity:

$$(C) \quad n_{k+1} - n_k \geq n_k - n_{k-1}.$$

*Proof of Lemma 3.1.* Let us prove first that (I) implies (II). We will prove the following a little stronger statement than (II)

$$(3.2) \quad \sum_{k=1}^{\infty} (n_{k+1} - n_k)^{1/2} n_k^{-1} |a_{n_k}| \sum_{j=1}^{n_k} |a_j| < \infty.$$

It is known (see [Z], Ch.1, S.9) that  $\{a_j\}_{j=1}^\infty \in l_2$  implies that

$$(3.3) \quad \{b_n\}_{n=1}^\infty \in l_2 \quad \text{with} \quad b_n := \frac{1}{n} \sum_{j=1}^n |a_j|.$$

We observe first that

$$(3.4) \quad \sum_{k=1}^{\infty} (n_{k+1} - n_k) b_{n_k}^2 < \infty.$$

Indeed, for any  $m > n$  we have

$$mb_m \geq nb_n$$

and for  $n_k < m < n_{k+1}$  we have for  $\{n_k\}_{k=1}^\infty \in \mathcal{M}$

$$b_{n_k} \leq \frac{n_{k+1}}{n_k} b_m \leq C b_m$$

with a constant  $C$  independent of  $k$  and  $m$ . Therefore

$$(3.5) \quad (n_{k+1} - n_k) b_{n_k}^2 \leq C^2 \sum_{m=n_k}^{n_{k+1}-1} b_m^2.$$

Combining (3.3) and (3.5) we get (3.4).

We return to (3.2)

$$\begin{aligned} \sum_{k=1}^{\infty} (n_{k+1} - n_k)^{1/2} n_k^{-1} |a_{n_k}| \sum_{j=1}^{n_k} |a_j| &= \sum_{k=1}^{\infty} (n_{k+1} - n_k)^{1/2} |a_{n_k}| b_{n_k} \leq \\ & \left( \sum_{k=1}^{\infty} a_{n_k}^2 \right)^{1/2} \left( \sum_{k=1}^{\infty} (n_{k+1} - n_k) b_{n_k}^2 \right)^{1/2} < \infty. \end{aligned}$$

Let us prove now that (II) implies (I). Assume the contrary that for  $\{n_k\}_{k=1}^{\infty} \in \mathcal{M}$  we have

$$(3.6) \quad \sum_{k=1}^{\infty} (n_{k+1} - n_k)^{1/2} n_k^{-1} < \infty.$$

We will construct a sequence  $\{a_j\}_{j=1}^{\infty} \in l_2$  such that

$$(3.7) \quad \liminf_{k \rightarrow \infty} a_{n_k} \sum_{j=1}^{n_k} a_j > 0.$$

Define

$$a_{n_k} := (n_{k+1} - n_k)^{1/4} n_k^{-1/2}; \quad a_j := (n_{k+1} - n_k)^{-1/4} n_k^{-1/2},$$

for

$$j \in (n_k, n_{k+1}), \quad k = 1, 2, \dots$$

Then (3.6) implies that  $\{a_j\}_{j=1}^{\infty} \in l_2$ . We have from the definition of  $\{a_j\}_{j=1}^{\infty}$  that

$$\sum_{j \in [n_k, n_{k+1})} a_j \geq (n_{k+1} - n_k)^{3/4} n_k^{-1/2}.$$

Next, we obtain from here

$$\begin{aligned} \sum_{j=1}^{n_{k+1}} a_j &\geq \sum_{l=1}^k \sum_{j \in [n_l, n_{l+1})} a_j \geq \sum_{l=1}^k (n_{l+1} - n_l)^{3/4} n_l^{-1/2} \geq \\ &\sum_{l=1}^k (n_{k+1} - n_k)^{-1/4} (n_{l+1} - n_l) n_l^{-1/2} \geq (n_{k+1} - n_k)^{-1/4} \sum_{l=1}^k (n_{l+1}^{1/2} - n_l^{1/2}) = \\ &(n_{k+1} - n_k)^{-1/4} (n_{k+1}^{1/2} - n_1^{1/2}). \end{aligned}$$

This estimate and the definition of  $a_{n_k}$  implies (3.7). Lemma 3.1 is proved now.



**Lemma 3.2.** Assume that a sequence  $\{n_k\}_{k=1}^{\infty} \in \mathcal{M}$  satisfies (I). Let  $0 < t \leq 1$  and  $\tau = \{t_n\}_{n=1}^{\infty}$  satisfies (3.1). Then  $\{f_{n_k-1}^{\tau}\}_{k=1}^{\infty}$  converges.

This lemma combined with the following simple modification of Lemma 2.1 from [T] give the sufficient part of the conclusion of Theorem 2.

**Lemma 3.3.** Assume that for some  $\{n_k\}_{k=1}^{\infty}$

$$\sum_{k=1}^{\infty} t_{n_k}^2 = \infty.$$

Then if  $\{f_{n_k-1}^{\tau}\}_{k=1}^{\infty}$  converges it converges to zero.

*Proof of Lemma 3.2.* This proof is similar to the corresponding arguments from [T]. We present it here for selfcompleteness of this paper. It is easy to derive from the definition of WGA the following two relations

$$(3.8) \quad f_m^{\tau} = f - \sum_{j=1}^m \langle f_{j-1}^{\tau}, \varphi_j^{\tau} \rangle \varphi_j^{\tau},$$

$$(3.9) \quad \|f_m^{\tau}\|^2 = \|f\|^2 - \sum_{j=1}^m |\langle f_{j-1}^{\tau}, \varphi_j^{\tau} \rangle|^2.$$

Denote  $a_j := |\langle f_{j-1}^{\tau}, \varphi_j^{\tau} \rangle|$ . We get from (3.9) that

$$\sum_{j=1}^{\infty} a_j^2 \leq \|f\|^2.$$

We take any two indices  $n < m$  and consider

$$\|f_n^{\tau} - f_m^{\tau}\|^2 = \|f_n^{\tau}\|^2 - \|f_m^{\tau}\|^2 - 2\langle f_n^{\tau} - f_m^{\tau}, f_m^{\tau} \rangle.$$

Denote

$$\theta_{n,m}^{\tau} := |\langle f_n^{\tau} - f_m^{\tau}, f_m^{\tau} \rangle|.$$

Using (3.8) and the definition of the WGA we get for all  $n < m$  that

$$(3.10) \quad \theta_{n,m}^{\tau} \leq \sum_{j=n+1}^m |\langle f_{j-1}^{\tau}, \varphi_j^{\tau} \rangle| |\langle f_m^{\tau}, \varphi_j^{\tau} \rangle| \leq \frac{a_{m+1}}{t_{m+1}} \sum_{j=1}^{m+1} a_j.$$

Specifying  $n = n_l - 1$  and  $m = n_k - 1$  we get from (3.10) that for any  $l < k$

$$(3.11) \quad \theta_{n_l-1, n_k-1}^{\tau} \leq t^{-1} a_{n_k} \sum_{j=1}^{n_k} a_j.$$

The relation (3.11) and Lemma 3.1 imply that

$$\lim_{k \rightarrow \infty} \max_{l < k} \theta_{n_l-1, n_k-1}^{\tau} = 0.$$

It remains to use the following simple lemma (see [T]).

**Lemma 3.4.** *Let in a Banach space  $X$  a sequence  $\{x_n\}_{n=1}^\infty$  be given. Assume that for any  $k, l$  we have*

$$\|x_k - x_l\|^2 = y_k - y_l + \theta_{k,l},$$

*with  $\{y_n\}_{n=1}^\infty$  is a convergent sequence of real numbers and  $\theta_{k,l}$  satisfying the property*

$$\lim_{l \rightarrow \infty} \max_{k < l} \theta_{k,l} = 0.$$

*Then  $\{x_n\}_{n=1}^\infty$  converges.*

We proceed now to the necessity part of Theorem 2. We will need the following simple properties of sequences from  $\mathcal{M}$ . Denote

$$\Delta n_k := n_{k+1} - n_k.$$

Then by monotonicity of  $\{\Delta n_k\}$  we have

$$(3.12) \quad n_{2^s} - n_{2^{s-1}} = \sum_{k=2^{s-1}}^{2^s-1} \Delta n_k \leq 2^{s-1} \Delta n_{2^s}$$

and

$$n_{2^s} - n_{2^{s-1}} \geq \sum_{k=1}^{2^{s-1}-1} \Delta n_k = n_{2^{s-1}} - n_1,$$

$$(3.13) \quad n_{2^s} - n_{2^{s-1}} \geq (n_{2^s} - n_1)/2.$$

Combining (3.12) and (3.13) we get

$$\Delta n_{2^s} \geq 2^{-s}(n_{2^s} - n_1)$$

and

$$(3.14) \quad (\Delta n_{2^s})^{-1/2} \leq 2^s (\Delta n_{2^s})^{1/2} (n_{2^s} - n_1)^{-1}.$$

Next,  $\{(\Delta n_k)^{1/2}/n_k\}$  is a monotone sequence:

$$(\Delta n_k)^{1/2}/n_k = n_k^{-1/2} (\Delta n_k/n_k)^{1/2}, \quad n_k \uparrow, \quad \Delta n_k/n_k \downarrow.$$

Thus the following two conditions are equivalent

$$(3.15) \quad \sum_{k=1}^{\infty} \frac{(\Delta n_k)^{1/2}}{n_k} < \infty,$$

$$\sum_{s=0}^{\infty} 2^s \frac{(\Delta n_{2^s})^{1/2}}{n_{2^s}} < \infty.$$

It is clear that (3.14) and (3.15) imply

$$\sum_{s=0}^{\infty} (\Delta n_{2^s})^{-1/2} < \infty.$$

The construction of the corresponding counterexample is similar to that from Section 2. We assume that

$$\sum_{s=0}^{\infty} (\Delta n_{2^s})^{-1/2} < \epsilon$$

with small enough  $\epsilon$ , say,  $\epsilon = 3/16$ . Define a new sequence  $\tau' := \{t'_n\}_{n=1}^{\infty}$  with

$$t'_n = (\Delta n_{2^s})^{-1/2}, \quad n \in [2^s, 2^{s+1}), \quad s = 0, 1, \dots$$

Then  $t'_n \downarrow 0$  and the WGA from Section 2 with  $\tau'$  and  $f := e_1$  reduces the square of the norm of  $f$  by at most  $2\epsilon$ . We modify now the above WGA. At each step  $n_k$  we replace WGA by Pure Greedy Algorithm (PGA) what means that at each step  $n_k$  we throw away a term  $c_l e_j$  with, say, the biggest  $j$ . Let us estimate how much do we reduce the square of the norm of  $f$  in this way. After  $s$  iterations we have

$$f^s = c_s(e_1 + \dots + e_{N_s}), \quad N_s \leq 2^s, \quad (c_s)^2 \leq 2^{-s}.$$

Relation (2.2) implies, that working on  $(s+1)$ -st iteration we will make at most  $N_s/t'_{2^s}$  steps of the algorithm. It is easy to see that after  $s$  iterations we have made at least  $2^s$  steps. Therefore we will use PGA at most

$$\frac{N_s}{t'_{2^s} \Delta n_{2^s}} + 1 \leq \frac{2^s}{(\Delta n_{2^s})^{1/2}} + 1$$

times during  $(s+1)$ -st iteration. Thus the norm  $\|\cdot\|^2$  will be reduced by at most

$$\frac{1}{(\Delta n_{2^s})^{1/2}} (1 + 2^{-s})$$

what gives the total reduction due to PGA steps at most  $2\epsilon$ . This reduction combined with the reduction of WGA at other steps sums up to at most  $4\epsilon$ . Choosing  $\epsilon$  small enough (say  $\epsilon = 3/16$ ) we get divergence of the defined above WGA.

**Remark 3.2.** In the proof of the necessity part of Theorem 2 we have used the convexity property (C) and also the monotonicity of  $\{(\Delta n_k)^{1/2}/n_k\}$  what is weaker than the assumption  $\{n_k\}_{k=1}^{\infty} \in \mathcal{M}$ .

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